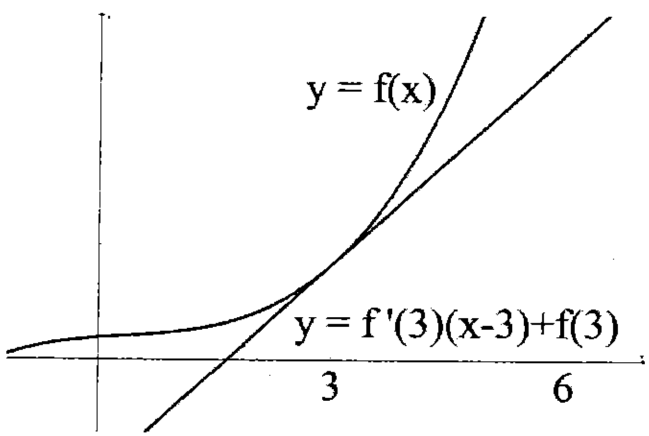
Taylor Notes 1 (TN 1):
Tangent Line Error Bounds
Goal: Approximate functions with tangent lines and get error bounds.

Def'n: The **first Taylor polynomial for f(x) based at b** is

$$T_1(x) = f(b) + f'(b)(x-b)$$



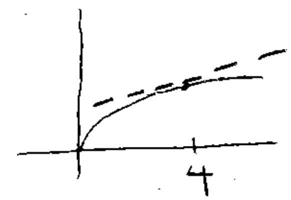
Entry Task: Find the 1st Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 4$.

- Use it to estimate $\sqrt{4.5}$.
- Use your calculator to find the difference between this estimate and the actual value.

$f(x) = \sqrt{x} \Rightarrow f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$T_1(x) = 2 + \frac{1}{4}(x-4)$



For $x \approx 4$, we have

$\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$

Thus

$\underbrace{\sqrt{4.5}}_{2.12132} \approx 2 + \frac{1}{4}(4.5-4) = 2.125$

$\text{ERROR} = |f(4.5) - T_1(4.5)|$

$= |\sqrt{4.5} - 2.125|$

$= |-0.0036796| = \underline{0.00}36796$

CORRECT TO TWO DIGITS!

Bounding the Error

Given an interval around $x = b$ (*i.e.* $b - a \le x \le b + a$).

Tangent Linear Error Bound Thm

If $|f''(x)| \le M$ for all *x*, then

$$|f(x) - T_1(x)| \le \frac{M}{2}|x - b|^2.$$

To use

Step 1: Find $f''(t)$.

Step 2: Find an upper bound (max) for |f''(t)| on the interval. Put this in for M in thm.

Step 3: Plug in x = "an endpoint" to get *worst case* error bound.

Example: Using the theorem, give an error bound for $|\sqrt{x} - T_1(x)|$ based at $x = 4$ on the interval $[3.5, 4.5]$.

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$\Rightarrow f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x^{3/2}}$$

$$|f''(x)| = \frac{1}{4x^{3/2}} \leftarrow \text{DECREASING FUNCTION, MAX MUST BE AT } 3.5$$

WHAT IS THE MAXIMUM FOR $3.5 \le x \le 4.5$

$$|f''(x)| \le \frac{1}{4(3.5)^{3/2}} = 0.03818 = M$$

$$|\sqrt{x} - T_1(x)| \le \frac{0.03818}{2}|x - 4|^2$$

$\uparrow$ 3.5 OR 4.5 $\left\{ \text{DOESN'T MATTER} \right.$

$$\frac{0.03818}{2}0.5^2 = 0.0047725$$

ON THIS INTERVAL THE ERROR IS NEVER BIGGER THAT 0.0047725

Example: $f(x) = \ln(x)$ at $b = 1$.

(a) Find the 1st Taylor polynomial.

(b) Use the error bound formula to find a bound on the error over the interval J = [1/2, 3/2]

(c) Find an interval around b = 1 where the error is less than 0.01.

x	$f(x)$	$T_1(x)$	$\lvert f(x) - T_1(x)\rvert$
1	0	0	0
1.2	0.1823	0.2	0.01768
1.4	0.3364	0.4	0.06353
0.9	-0.1053	-0.1	0.00536

(a) $f(x) = \ln(x) \Rightarrow f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$

$T_1(x) = 0 + 1(x-1) = x-1$

$\boxed{\ln(x) \approx x - 1 \quad \text{for} \quad x \approx 1}$

(b) $f''(x) = -\frac{1}{x^2}$

$|f''(x)| = \frac{1}{x^2}$ ← DECREASING FUNCTION MAX WILL BE AT 1/2

$\frac{1}{x^2} \leq \frac{1}{(1/2)^2} = 4 = M$

$|\ln(x) - (x-1)| \leq \frac{4}{2}|x-1|^2 \leq 2(0.5)^2 = 0.5$ ← EITHER ENDPOINT

(c) WANT $[1-a, 1+a]$

SUCH THAT

$|f(x) - T_1(x)| \leq \frac{M}{2}|x-1|^2 \leq 0.01$

ON $[1-a, 1+a]$

$|f''(x)| \leq \frac{1}{(1-a)^2} = M$

$\frac{M}{2}|x-1|^2 = \frac{1}{2}\frac{1}{(1-a)^2}(1+a-1)^2$

$= \frac{1}{2}\frac{a^2}{(1-a)^2} \overset{?}{=} 0.01$

$\left(\frac{a}{1-a}\right)^2 = 0.02$

$\frac{a}{1-a} = \sqrt{0.02}$

$a = \sqrt{0.02}(1-a)$

$(1+\sqrt{0.02})a = \sqrt{0.02}$

$a = \frac{\sqrt{0.02}}{1+\sqrt{0.02}} \simeq 0.1239$

$[1 - 0.1239, 1 + 0.1239]$

Proof of error bound for x > b:

Start with $f(x) - f(b) = \int_b^x f'(t)dt$.

Do integration by parts,

(with $u = f'(t)$, $dv = dt$, CAN BE ANY CONSTANT, WE CHOOSE $-x$ TO MAKE ANSWER COME OUT NICER

$du = f''(t)$, $v = t - x$)

to get

$$f(x) - f(b) = f'(b)(x-b) - \int_b^x (t-x)f''(t)dt$$

Rearrange to get

$$f(x) - f(b) - f'(b)(x-b) = \int_b^x (x-t)f''(t)dt$$

Thus,

$$|f(x) - T_1(x)| = \left|\int_b^x (x-t)f''(t)dt\right|$$

Then note

$$\left|\int_b^x (x-t)f''(t)dt\right| \le \int_b^x (x-t)|f''(t)|dt$$

UPPER BOUND

$$\le M\int_b^x (x-t)dt$$

$$\le \frac{M}{2}(x-b)^2.$$

Note about "Bounds":

An upper **bound**, *M*, is a number that is always bigger than the function. The smallest possible upper bound is sometimes called a *tight* bound.

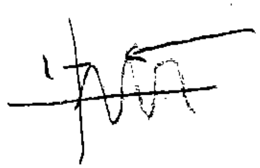
Examples: Find any upper **bound** (if it is easy to do so, find a *tight* upper bound).

1. | $\sin(5x)$ | on [0,2π]

$$|\sin(5x)| \leq 1$$

↑

"TIGHT" UPPER BOUND



2. $|x-3|$ on [1,5]

$$|x-3| \leq 2$$

TIGHT

3. $\left|\frac{1}{(2-x)^3}\right|$ on [-1,1]

$$\left|\frac{1}{(2-x)^3}\right|$$

WILL BE LARGEST WHEN DENOMINATOR IS SMALLEST WHICH IS AT $x = 1$

$$\left|\frac{1}{(2-x)^3}\right| \leq \frac{1}{(2-1)^3} = 1$$

4. $|\sin(x) + \cos(x)|$ on $[0,2\pi]$

NOT TIGHT

$|\sin(x) + \cos(x)| \le 1 + 1 = 2$

TIGHT UPPER BOUND
IS HARDER TO FIND HERE

$|\sin(x) + \cos(x)| \le \sqrt{2}$

5. $\left|\cos(2x) + e^{2x} + \frac{6}{x}\right|$ on $[1,4]$

$\left|\cos(2x) + e^{2x} + \frac{6}{x}\right|$

NOT TIGHT

$\le 1 + e^{8} + 6 = 7 + e^{8}$

HARDER TO GET TIGHT UPPER BOUND

Example (you do):

Let $f(x) = x^{1/3}$ and b = 8.

(a) Find the 1st Taylor Polynomial.

(b) Give a bound on the error over the interval J = [7,9].

$$f(x) = x^{1/3} \Rightarrow f(8) = 8^{1/3} = 2$$

$$f'(x) = \frac{1}{3} x^{-2/3} \Rightarrow f'(8) = \frac{1}{3} \frac{1}{8^{2/3}} = \frac{1}{3} \frac{1}{2^2} = \frac{1}{12}$$

$$T_1(x) = 2 + \frac{1}{12}(x-8)$$

$$x^{1/3} \approx 2 + \frac{1}{12}(x-8) \quad \text{for } x \approx 8$$

$$f''(x) = -\frac{2}{9} x^{-5/3} = -\frac{2}{9x^{5/3}}$$

DECREASING FUNCTION

$$|f''(x)| \leq \frac{2}{9x^{5/3}} \leq \frac{2}{9(7)^{5/3}}$$

MAX AT $x = 7$

$$\approx 0.008675 = M$$

$$|f(x) - T_1(x)| \leq \frac{M}{2}|x-8|^2$$

$$\frac{1}{2}(0.008675)|9-8|^2$$

$$= 0.0043377$$

Ex

$$\sqrt[3]{9} \approx 2 + \frac{1}{12}(9-8)$$

$$2 + \frac{1}{12} = 2.08\overline{3}$$

ACTUAL $= 2.0800838$